

Deleting a marked item from an unsorted database with a single query

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In this Letter we present a quantum deletion algorithm that deletes a marked state from an unsorted database of N items with only a single query. This algorithm achieves exponential speedup compared with classical algorithm where $O(N)$ number of query is required. General property of this deleting algorithm is also studied.

The merging of quantum mechanics and information theory has been very fruitful in recent decades. A quantum computer can complete difficult tasks that is hard for a classical computer. Remarkably, Shor have shown that quantum computer can factorize a large integer exponentially faster than a classical computer [1], and Grover has proved that quantum computer can search an unsorted database with a square-root speedup compared [2]. Searching an unsorted database is a widely met scientific problem [3]. There have been extensive studies on the generalization of the Grover algorithm [4, 5, 6, 7, 8, 9, 10].

A related task is to delete an item from a database. In a chain data structure, if it is known that certain nodes does not satisfy the data structure conditions or does not meet the algorithm demands, then one should delete them from database for convenient searching and visiting. For instance, in the PageRank algorithm [11] where the web pages were sorted according to their click amounts, we usually select the former N items which are clicked most frequently from the magnanimity of information. Therefore we need to add or delete the web pages with respect to the marked node in order to dynamically preserve the former N nodes. Then the former N nodes form a heap structure which requires kinds of operations: addition, deletion, search and visit. In some case, finding a marked item from an unsorted database is prerequisite step for its next disposal step. For example, if a database represents the components of a giant machine, and we want to find out the bad component and delete it from the database. Usually, deleting a marked item from an unsorted database with N items is equivalent to finding a marked state from the database, and it requires $O(N)$ steps in classical computing.

In this Letter, we present a quantum algorithm that deletes an marked item from an unsorted database with only a single query. Compared to its classical counterpart, it achieves an exponential speedup. At first it may sounds very similar to quantum searching. But, actually, it is very different. Indeed, here we do not require the knowledge of the marked state: what counts is to delete it from the database.

The abstract problem is: if there is an unsorted database with $N = 2^n$ items, one item satisfies a query function $f(\tau) = 1$, and all other items satisfy $f(x) = 0$, the task is to delete the item τ from the database.

In quantum mechanics, the problem becomes the following, in the evenly superposed state $|\psi_0\rangle$, which represents the database consisting of all the items

$$|\psi_0\rangle = \sqrt{\frac{1}{N}} (|0\rangle + |1\rangle + \cdots + |\tau\rangle + \cdots + |N-1\rangle) \quad (1)$$

one item τ satisfies the query $f(\tau) = 1$, and all other items satisfy $f(x) = 0$. The task is to delete the item τ from the superposed state (1). Let $|c\rangle$ indicates a state that is the sum over all i which are not the marked state,

$$|c\rangle = \sqrt{\frac{1}{N}} \sum_{i \neq \tau} |i\rangle. \quad (2)$$

In the two-dimensional spaces span by $|c\rangle$ and $|\tau\rangle$, the initial state of the quantum computer $|\psi_0\rangle$ may be re-expressed as follows

$$|\psi_0\rangle = \cos \beta |c\rangle + \sin \beta |\tau\rangle, \quad (3)$$

where the coefficients

$$\sin \beta = \sqrt{\frac{1}{N}}, \quad \cos \beta = \sqrt{\frac{N-1}{N}}. \quad (4)$$

The quantum deletion algorithm consists of successive applications of a quantum deletion subroutine, indicated as S operation. The S operation contains four steps:

Step 1: Perform a conditional phase shift $e^{i\phi}$ to every computational basis state except the marked state $|\tau\rangle$, the action of this step is denoted as I_c

$$I_c = I + (e^{i\phi} - 1) \sum_{i \neq \tau} |i\rangle\langle i|. \quad (5)$$

Step 2: Perform the n -qubit Hadamard transform W , namely $W = H \otimes^n$, where H is the Hadamard transform on a single qubit.

Step 3: Perform a conditional phase shift $e^{i\phi}$ to the $|0\rangle$ state and leaves all other basis states untouched. Denoting this action as I_0 , and it is

$$I_0 = I + (e^{i\phi} - 1)|0\rangle\langle 0|. \quad (6)$$

Step 4: Perform the n -qubit Hadamard transform again.

The deletion operator S can be represented in a matrix form in the spaces span by $|c\rangle$ and $|\tau\rangle$,

$$\begin{aligned} S &= -WI_0WI_c \\ &= \begin{bmatrix} -e^{i\phi}(1 + (e^{i\phi} - 1)\cos^2\beta) & -(e^{i\phi} - 1)\sin\beta\cos\beta \\ -e^{i\phi}(e^{i\phi} - 1)\sin\beta\cos\beta & -e^{i\phi} + (e^{i\phi} - 1)\cos^2\beta \end{bmatrix}. \end{aligned} \quad (7)$$

In the above procedures, the two phases are equal, and this is deeply rooted in the phase matching condition in quantum search algorithm [8, 9, 10]. We have worked out the explicitly the phase ϕ ,

$$\phi = 2 \arcsin\left(\frac{1}{2\cos\beta}\right). \quad (8)$$

When applying S to the database in Eq.(1), through direct calculation we obtain,

$$|\psi_1\rangle = e^{i(\phi-\pi)/2} \sqrt{\frac{1}{N-1}} (|0\rangle + |1\rangle + \cdots + |N-1\rangle) \quad (9)$$

That is to say, except a global phase factor $(\phi - \pi)/2$, the marked state $|\tau\rangle$ has been deleted from the unsorted database with certainty using only one application of the deletion operation S . The phase factor can be eliminated by applying a phase rotation of $e^{(\pi-\phi)/2}$ to all the basis state. It is striking that quantum deletion algorithm can delete a marked item from an unsorted database with just a single query, which is ultimate performance of any deletion algorithm. In comparison, a classical computer requires $O(N)$ queries to find the marked state and then delete it from the database.

Repeating the deletion operation for a number of times, we find the deletion algorithm will still be effective for certain times of iteration. We investigate the periodic property of quantum deletion algorithm and calculate the performance of k times of S iteration. Firstly, we can write out the expression of S operator in a diagonalized form

$$S = U\Lambda U^\dagger, \quad (10)$$

where U is

$$\sqrt{\frac{1}{R}} \begin{bmatrix} e^{-\frac{i\phi}{2}}(\cos\frac{\phi}{2}\cos\beta + \cos\beta') & -\sin\beta \\ \sin\beta & e^{\frac{i\phi}{2}}(\cos\frac{\phi}{2}\cos\beta + \cos\beta') \end{bmatrix}, \text{ is } \quad \text{and } \theta = 2k\beta' = \frac{k\pi}{3}. \quad \text{The final analytic expression for } S^k$$

$$S^k = (-1)^k e^{ik\phi} \begin{bmatrix} \cos\theta + i\sin\theta\sqrt{\frac{3N-4}{3N}} & \sin\theta\sqrt{\frac{1}{3(N-1)}} + i\sin\theta\sqrt{\frac{3N-4}{3N(N-1)}} \\ -\sin\theta\sqrt{\frac{1}{3(N-1)}} + i\sin\theta\sqrt{\frac{3N-4}{3N(N-1)}} & \cos\theta - i\sin\theta\sqrt{\frac{3N-4}{3N}} \end{bmatrix}. \quad (18)$$

As shown in Table. I, functions $\sin\theta$ and $\cos\theta$ vary periodically with a period of 6 in k , and functions of

$$\Lambda = \begin{bmatrix} -e^{i(\phi+2\beta')} & 0 \\ 0 & -e^{i(\phi-2\beta')} \end{bmatrix}, \quad (11)$$

$$\beta' = \arcsin(\sin\frac{\phi}{2}\cos\beta), \quad (12)$$

$$R = \sin^2\beta + (\cos\frac{\phi}{2}\cos\beta + \cos\beta'). \quad (13)$$

Using Eq. (4) and noting that

$$\sin\frac{\phi}{2} = \frac{1}{2}\sqrt{\frac{N}{N-1}}, \quad \cos\frac{\phi}{2} = \frac{1}{2}\sqrt{\frac{3N-4}{N-1}}, \quad (14)$$

we obtain

$$\begin{aligned} \beta' &= \arcsin\frac{1}{2} = \frac{\pi}{6}, \\ R &= \frac{3N + \sqrt{3N(3N-4)}}{2N}. \end{aligned} \quad (15)$$

The k successive operations of S can be written analytically

$$S^k = U\Lambda^k U^\dagger = \begin{bmatrix} s_{11} & s_{12} \\ s_{21} & s_{22} \end{bmatrix}, \quad (16)$$

where

$$\begin{aligned} s_{11} &= (-1)^k e^{ik\phi} (\cos\theta + i\sin\theta\sqrt{\frac{3N-4}{3N}}), \\ s_{12} &= (-1)^k e^{ik\phi} [\sin\theta\sqrt{\frac{1}{3(N-1)}} + i\sin\theta\sqrt{\frac{3N-4}{3N(N-1)}}], \\ s_{21} &= (-1)^k e^{ik\phi} [-\sin\theta\sqrt{\frac{1}{3(N-1)}} + i\sin\theta\sqrt{\frac{3N-4}{3N(N-1)}}], \\ s_{22} &= (-1)^k e^{ik\phi} (\cos\theta - i\sin\theta\sqrt{\frac{3N-4}{3N}}), \end{aligned} \quad (17)$$

$(-1)^k \sin\theta$ and $(-1)^k \cos\theta$ vary periodically with a period 3. Hence S^k is a periodic function of k with 3 as its

period. We now look at the three different situations.

Case 1. When $k = 3m + 1$, i.e. $k = 1, 4, 7, 10, 13 \dots$, Eq. (18) reduces to the following form

$$S^k = e^{ik\phi} \begin{bmatrix} -\frac{1}{2} - \frac{i}{2}\sqrt{\frac{3N-4}{N}} & -\frac{1}{2}\sqrt{\frac{1}{N-1}} - \frac{i}{2}\sqrt{\frac{3N-4}{N(N-1)}} \\ \frac{1}{2}\sqrt{\frac{1}{N-1}} - \frac{i}{2}\sqrt{\frac{3N-4}{N(N-1)}} & -\frac{1}{2} + \frac{i}{2}\sqrt{\frac{3N-4}{N}} \end{bmatrix}. \quad (19)$$

After S^k , the state becomes

$$\begin{aligned} |\psi_k\rangle &= S^k(\cos\beta|c\rangle + \sin\beta|\tau\rangle) \\ &= e^{i[(k-1/2)\phi - \pi/2]} \sqrt{\frac{1}{N-1}}(|0\rangle + |1\rangle + \dots + |N-1\rangle). \end{aligned} \quad (20)$$

Hence this type of iteration process accomplishes the following transformation eventually

$$\sqrt{\frac{1}{N}} \sum_{i=0}^{N-1} |i\rangle \xrightarrow{S^{3m+1}} e^{i[(k-1/2)\phi - \pi/2]} \sqrt{\frac{1}{N-1}} \sum_{i \neq \tau} |i\rangle. \quad (21)$$

We can see that if the number of iteration is $k = 3m + 1$, where m is any nonnegative integer, after k deletion iterations the marked state $|\tau\rangle$ will be successfully deleted from the unsorted database. The global phase factor can be left alone, or by a simultaneous phase rotation of $e^{-i[(k-1/2)\phi - \pi/2]}$ to all basis states to eliminate the total phase.

Case 2. When $k = 3m + 2$, i.e. $k = 2, 5, 8, 11, 14 \dots$, Eq. (18) can be rewritten as

$$S^k = e^{ik\phi} \begin{bmatrix} -\frac{1}{2} + \frac{i}{2}\sqrt{\frac{3N-4}{N}} & \frac{1}{2}\sqrt{\frac{1}{N-1}} + \frac{i}{2}\sqrt{\frac{3N-4}{N(N-1)}} \\ -\frac{1}{2}\sqrt{\frac{1}{N-1}} + \frac{i}{2}\sqrt{\frac{3N-4}{N(N-1)}} & -\frac{1}{2} - \frac{i}{2}\sqrt{\frac{3N-4}{N}} \end{bmatrix}. \quad (22)$$

After k times of S iteration, the initial evenly distributed state becomes

$$|\psi_k\rangle = S^k \begin{bmatrix} \cos\beta \\ \sin\beta \end{bmatrix} = e^{ik\phi} \begin{bmatrix} e^{i(\pi+\phi)} \sqrt{\frac{N-1}{N}} \\ e^{i\pi} \sqrt{\frac{1}{N}} \end{bmatrix}, \quad (23)$$

where

$$e^{i(\pi+\phi)} = -\frac{N-2}{2(N-1)} + \frac{i\sqrt{N(3N-4)}}{2(N-1)}. \quad (24)$$

This type of transformation can be represented consequently as

$$\sqrt{\frac{1}{N}} \sum_{i=0}^{N-1} |i\rangle \xrightarrow{S^{3m+2}} \frac{e^{i[\pi+(k+1)\phi]}}{\sqrt{N}} \sum_{i \neq \tau} |i\rangle + \frac{e^{i(\pi+k\phi)}}{\sqrt{N}} |\tau\rangle. \quad (25)$$

It implies that after $k = 3m + 2$ number of iterations the marked state can not be deleted except for a different phase change of the marked state relative to other states.

Case 3. When $k = 3m + 3$, i.e. $k = 3, 6, 9, 12, 15 \dots$, we may reduce Eq. (18) to

$$S^k = e^{ik\phi} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = e^{ik\phi} I, \quad (26)$$

so except a global phase factor $e^{ik\phi}$, k times of iteration S leaves the state of the system state in the evenly distributed state.

$$|\psi_k\rangle = S^k \begin{bmatrix} \cos\beta \\ \sin\beta \end{bmatrix} = e^{ik\phi} \begin{bmatrix} \sqrt{\frac{N-1}{N}} \\ \sqrt{\frac{1}{N}} \end{bmatrix}. \quad (27)$$

This type of iteration process can be expressed as

$$\sqrt{\frac{1}{N}} \sum_{i=0}^{N-1} |i\rangle \xrightarrow{S^{3m+3}} e^{ik\phi} \frac{1}{\sqrt{N}} \sum_{i=0}^{N-1} |i\rangle. \quad (28)$$

We can see if the number of iteration is $k = 3m + 3$, the initial evenly distributed state of the unsorted database would remain.

Next we present an approximate deletion algorithm that uses a fixed phase rotation. In the unsorted database search problem, the Grover's algorithm which finds a marked item with high probability, whereas the Long algorithm finds the marked item with certainty by using datasize dependent phase rotations [7]. A schematic plot for the phase angle ϕ versus database size is given in Fig. 1. The approximate deletion algorithm replaces the

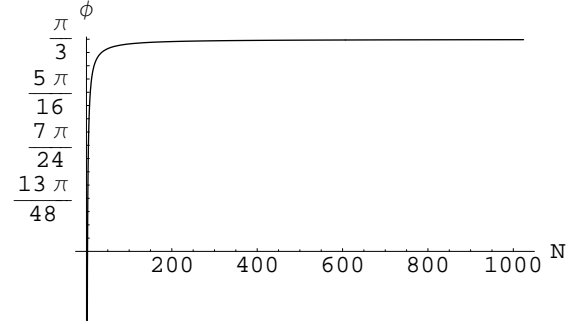


FIG. 1: The phase angle ϕ versus the datasize N .

datasize phase angle with a constant phase angle, namely

$$\phi_0 = \lim_{N \rightarrow \infty} \phi = \frac{\pi}{3}, \quad (29)$$

that is, steps 1 and 3, the phase rotations are replaced by phase rotations through $\pi/3$. Then the S matrix becomes

$$S = \begin{bmatrix} \frac{N-2}{2N} - i\frac{\sqrt{3}}{2} & \frac{\sqrt{N-1}}{2N} - i\frac{\sqrt{3(N-1)}}{2N} \\ \frac{\sqrt{N-1}}{N} & \frac{-2N+1}{2N} - i\frac{\sqrt{3}}{2N} \end{bmatrix}. \quad (30)$$

After the operation of S , the norm of the component of the marked state becomes $N^{-3/2}$

$$\lim_{N \rightarrow \infty} \frac{1}{N^{3/2}} = 0. \quad (31)$$

Therefore under the large datasize limit, the marked state component approaches zero, or say, the marked state is near completely deleted.

TABLE I: periodic property of some trigonometric functions

k	1	2	3	4	5	6	7	8	9	10	11	12
$\sin \theta$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{2}$	0	$-\frac{\sqrt{3}}{2}$	$-\frac{\sqrt{3}}{2}$	0	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{2}$	0	$-\frac{\sqrt{3}}{2}$	$-\frac{\sqrt{3}}{2}$	0
$\cos \theta$	$\frac{1}{2}$	$-\frac{1}{2}$	-1	$-\frac{1}{2}$	$\frac{1}{2}$	1	$\frac{1}{2}$	$-\frac{1}{2}$	-1	$-\frac{1}{2}$	$\frac{1}{2}$	1
$(-1)^k \sin \theta$	$-\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{2}$	0	$-\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{2}$	0	$-\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{2}$	0	$-\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{2}$	0
$(-1)^k \cos \theta$	$-\frac{1}{2}$	$-\frac{1}{2}$	1	$-\frac{1}{2}$	$-\frac{1}{2}$	1	$-\frac{1}{2}$	$-\frac{1}{2}$	1	$-\frac{1}{2}$	$-\frac{1}{2}$	1

In summary, a quantum deletion algorithm with certainty is present. This algorithm deletes the marked state from an unsorted database. We have shown that the quantum deletion algorithm completes deletion with only a single query, in contrast to the $O(N)$ steps required in classical computing. Further it was discovered that the deletion operation is periodic and has period of 3. Moreover, we have present an approximate quantum deletion

algorithm in which the phase rotation has a fixed value of $\pi/3$.

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